Manual of Precalculus Mathematics

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Preface

This text is written for students preparing for a course in calculus for a career in business; economics or social sciences. Calculus is the initial study of limits, continuity, differentiation and integration. There are many techniques from elementary mathematics that are needed in calculus. This manual contains a review of such topics, including sets; real numbers; equations and inequalities; polynomials; exponents and logarithms; analytic geometry; functions and graphs. It focuses on all the skills that the student preparing for calculus will need in the future.

In this volume we present a selection of mathematical formulas often used in calculus, statistics and economics. It contains definitions and the main results of theory. This text focuses on a problem-solving approach to the subject and contains many exercises.

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Chapter 1

Fundamentals

1.1 Symbols

It is convenient to use some symbols and abbreviations for notions and logical expressions. The following is a list of the most important symbols that are used in this book.

- $P \Longrightarrow Q$ means "if P then Q" or "P implies Q"

Remark 1 *If $P \Longrightarrow Q$ then we say that P is a sufficient condition for Q and that Q is a necessary condition for P*

Remark 2 *$P \Longrightarrow Q$ is equivalent to not $Q \Longrightarrow$ not P*

- $P \Longleftrightarrow Q$ means "P if and only if Q" or "P implies Q and Q implies P"

Remark 3 *If $P \Longleftrightarrow Q$ then we say that P is a necessary and sufficient condition for Q*

- $x \in S$: the element x belongs to the set S.
- $x \notin S$: the element x does not belong to the set S.
- $\exists$: there exists
- $\exists!$: there exists a unique

- $\forall$: for all
- i.e.: that is
- iff: $\Longleftrightarrow$ (i.e., if and only if)
- $\emptyset$: empty set
- $\{\}$: the set of
- $\mathbb{N}$: the set $\{0, 1, 2, \ldots\}$ of natural numbers
- $\mathbb{N}^+$: the set $\{1, 2, \ldots\}$ of positive natural numbers
- $\mathbb{Z}$: the set $\{0, \pm 1, \pm 2, \ldots\}$ of integers
- $\mathbb{Z}^+$: the set of positive integers
- $\mathbb{Q}$: the set $\left\{\dfrac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$ of rational numbers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}^+$: the set of positive real numbers
- $\mathbb{R}^-$: the set of negative real numbers
- $\mathbb{R}^* = \mathbb{R} - \{0\}$
- $\mathbb{C}$: the set of complex numbers

1.2 Sets and subsets

In calculus, as in other areas of mathematics, the language of set theory is one of the most important tools. In this section we shall describe some of the notions from set theory that will be useful later.

The basic notions of set theory are those of set and the idea of membership of a set. We express this latter notion by $\in$, and we write

$$x \in A$$

for the statement "x is an element (or member) of A". A set is completely determined by its members; i.e., if two sets A and B have the property that

$$x \in A \Longleftrightarrow x \in B$$

then $A = B$. Since a set is determined by its elements, one of the commonest ways of determining a set is by specifying its elements as in the following definition: A is the set of all elements $x \in X$ which verify the property P. In symbols:

$$A = \{x \in X : \ x \text{ verifies } P\}$$

We usually think of a set as having some elements, but it is also convenient to consider the set that has no elements. We call it the *empty set* and we denote it by $\varnothing$.

If a, b, c are elements of X, we define the *unit* set $\{a\}$ to be the set whose only element is a. We also define $\{a, b\}$ to be the set whose elements are exactly a and b; $\{a, b, c\}$ to be the set whose elements are exactly a, b and c and so forth.

1.2.1 Subsets

Let A and B be any two sets. Then A is a *subset* of B if every member of A is also a member of B. In symbols:

$$A \subset B \Longleftrightarrow \forall x \in A, x \in B$$

Remark 4 $A = B \Longleftrightarrow A \subset B$ *and* $B \subset A$

Remark 5 $\varnothing \subset A, \forall A$ *(the empty set is a subset of every set)*

1.2.2 Set operations

If $A, B \subset \Omega$ (the *universal* set) then:

- $A \cup B = \{x : x \in A \text{ and } x \in B\}$ (A *union* B)
- $A \cap B = \{x : x \in A \text{ or } x \in B\}$ (A *intersection* B)
- $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$ (A *minus* B)
- $A^C = \Omega \setminus A = \{x \in \Omega : x \notin A\}$ (the *complement* of A)

1.2.3 Examples

1. $\{1,2,3\} \cup \{a,b,c\} = \{1,2,3,a,b,c\}$

2. $\{1,2,3\} \cup \{3,5\} \cup \{7\} = \{1,2,3,5,7\}$

3. $\{1,2,3\} \cap \{a,b,c\} = \emptyset$

4. $\{1,2,3\} \cap \emptyset = \emptyset$

5. $\{a,b,c,d\} \setminus \{d,e,f\} = \{a,b,c\}$

6. $\{1,2,3\} \setminus \{a,b,c\} = \{1,2,3\}$

7. $\{1,2,3\} \setminus \{1,2,3\} = \emptyset$

8. $(\{1,2,3,c\} \cap \{2,4,6\}) \cup (\{1,2,3,c\} \cap \{a,b,c\}) = \{2,c\}$

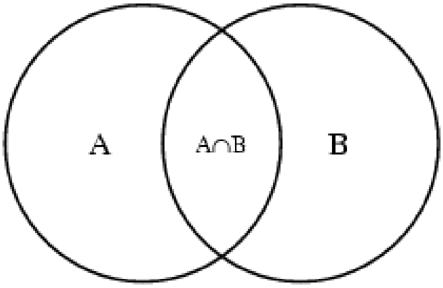
9. $(\{2,4,6\} \cup \{a,b,c\}) \setminus \{2,a,b\} = \{c,4,6\}$

1.2.4 Important properties

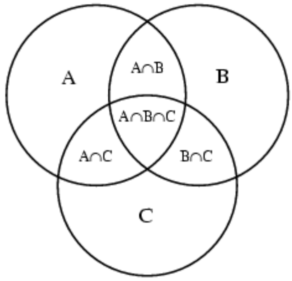
- $A \subset B \Longrightarrow B^C \subset A^C$
- $(A \cup B)^C = A^C \cap B^C$ (DeMorgan's law)
- $(A \cap B)^C = A^C \cup B^C$ (DeMorgan's law)
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **Power set:** If A is a set, the collection of subsets of A (which includes A and $\emptyset$) is denoted by 2^A or $\mathcal{P}(A)$ and is called the *power set* of A.

1.2.5 Venn diagrams

The Venn diagram is made up of two or more overlapping circles or rectangles. It is often used to show relationships between sets and in particular to depict set intersections The definition of an intersection of sets is illustrated as a Venn diagram in the following figures.



Venn diagram of $A \cap B$. The figure consists of two intersecting circles representing a total of three regions, A, B and $A \cap B$.



Venn diagram of $A \cap B \cap C$, $A \cap B$, $A \cap C$ and $B \cap C$

.

Venn diagrams are useful to derive formulas. For example, it is easy to prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ just by showing that the right- and left-hand side of the equality correspond to the same Venn representation.

1.3 Natural numbers

The numbers used for counting: $0, 1, 2, 3, ...$ are called *natural numbers*. They are often also referred to as non-negative integers. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of all natural numbers and by $\mathbb{N}^* = \mathbb{N} - \{0\}$ the set $\{1, 2, \ldots\}$ of all positive natural numbers.

Adding or multiplying natural numbers always produces other natural numbers. While it is in general not possible to divide one natural number by another and get a natural number as result, the procedure of division with remainder is available as a substitute: for any two natural numbers a and b with $b \neq 0$ we can find natural numbers q and r such that $a = bq + r$ and $r < b$. The number q is called the *quotient* and r is called the *remainder* of division of a by b. The numbers q and r are uniquely determined by a and b.

We say that b is a *divisor* (or *factor*) of an integer a if the remainder r of division of a by b is 0. We also say a is *divisible* by b or a is a *multiple* of b or b *divides* a and we usually write $a = \dot{b}$. For example, 9 is a *divisor* of 56 because $56 = 9.8$. The positive divisors of 42 are {1, 2, 3, 6, 7, 14, 21, 42}.