Computation of Business-Cycle Models with the Generalized Schur Method*

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Abstract:
We describe an algorithm that is able to compute the solution of a singular linear difference system under rational expectations. The algorithm uses the Generalized Schur Factorization and is illustrated by a simple example.

1 Introduction

Modern business cycle theory uses stochastic dynamic general equilibrium models in order to explain and forecast the behavior of economic variables such as income, employment, or inflation. In Heer and Maußner (2009), we provide a comprehensive review of both linear and non-linear computational methods in order to solve such models.

In most cases, business cycle models are solved with the help of log-linearization around the deterministic steady state. This method is very convenient for at least three reasons:

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1) This method is simple, fast, and easy to implement. 2) As shown by Aruoba et al. (2006) and Heer and Maußner (2008), log-linearization often provides for a very accurate approximation, in particular if one is interested in the statistical properties of the economic variables. And 3), the solution from this linear method can be used as an initial guess for the computation of a non-linear solution.

In general, the complex stochastic dynamic general equilibrium model of the business cycle can be log-linearized around the deterministic steady state resulting in the follow set of equations:[1]

$$C_u \mathbf{u}_t = C_{x\lambda} \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} + C_z \mathbf{z}_t, \tag{1a}$$

$$D_{x\lambda} \mathbb{E}_t \begin{bmatrix} \mathbf{x}_{t+1} \\ \boldsymbol{\lambda}_{t+1} \end{bmatrix} + F_{x\lambda} \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} = D_u \mathbb{E}_t \mathbf{u}_{t+1} + F_u \mathbf{u}_t + D_z \mathbb{E}_t \mathbf{z}_{t+1} + F_z \mathbf{z}_t, \tag{1b}$$

$$\mathbf{z}_{t+1} = \Pi \mathbf{z}_t + \boldsymbol{\epsilon}_{t+1}, \tag{1c}$$

$$\boldsymbol{\epsilon} \sim N(0, \Sigma). \tag{1d}$$

where $\mathbf{x}_t$ denotes the state variables in period t, $\mathbf{u}_t$ the control variables, $\boldsymbol{\lambda}_t$ the costate variables, and $\mathbf{z}_t$ the aggregate shocks. The variables in these vectors usually represent percentage deviations of the model's original variables from their respective stationary values. The matrices C and D are derived from the equilibrium conditions of the model. They are functions of the deep parameters of the model that, for example, describe the preferences of the households or the production function. The exogenous variables follow an autoregressive process characterized by the matrix Π. Expectations $\mathbb{E}_t$ are conditional upon the information in period t.

In general, the static equations (1a) and the exogenous progress (1c) can be substituted in (1b) in order to have dynamic equations in the state and costate variables and the exogenous shocks only:

$$B \mathbb{E}_t \begin{bmatrix} \mathbf{x}_{t+1} \\ \boldsymbol{\lambda}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} + C \mathbf{z}_t, \tag{2}$$

[1] The reader who is familiar with the method of log-linearization will notice the resemblance of the equations (1) and (2) with the equations (A40) and (A41) in King, Plosser and Rebelo (KPR) (2002). The algorithm of KPR, however, is not able to solve the kind of models that we consider in the following.

We will illustrate the derivation of theses matrices and the above equation in the following by means of a very simple example. In our example, however, the matrix B is not invertible.[2] This generates a problem for many of the existing algorithms that rely upon the decomposition of $B^{-1}A$ with the help of either the Jordan or the Schur factorization so that the computation breaks down in these cases. As an alternative, we will suggest the use of the Generalized Schur factorization instead.

2 A Simple Example: A Two Period Overlapping Generations (OLG) Model

2.1 The Model

There are two generations $s = 1, 2$ alive at period t. Each generation consists of a representative member of measure one. The young household works $n = 1$ hours and receives wage w_t. The old household consumes his savings. He earns interest r on the capital stock $k_{t+1,2}$ at age $s = 2$ in period $t + 1$. Capital depreciates at the rate δ. Thus, at t the young household solves

$$\begin{aligned} &\max \quad \ln c_{t,1} + \beta \mathbb{E}_t \ln c_{t+1,2}, \\ &\text{subject to} \\ &\quad c_{t,1} = w_t - k_{t+1,2}, \\ &c_{t+1,2} = (1 - \delta + r_{t+1}) k_{t+1,2}, \end{aligned} \tag{3}$$

where $c_{t,s}$ denotes consumption at age s in period t. Expectations $\mathbb{E}_t$ are rational and conditional on information at the beginning of period t.

Aggregate capital equals $K_t = k_{t,2}$ and aggregate labor amounts to $N = n = 1$. Output is produced with the help of labor N and capital K

$$Y_t = Z_t N^{1-\alpha} K_t^{\alpha}, \tag{4}$$

and is subject to a technology shock Z_t. We assume Z_t to follow a stationary stochastic

[2] McCandless (2008), Section 6.8.4, illustrates the Generalized Schur factorization with the indivisible labor model of Hansen (1985). Since he does not distinguish between costate and control variables his setup of the model implies a non invertible B matrix. Sims (1989) is an early application of the Generalized Schur factorization.

process.[3] Specifically, we assume that the percentage deviation of Z_t from its unconditional mean of $Z = 1$, $\hat{Z}_t := (Z_t - Z)/Z$, is governed by a first-order autoregressive process

$$\hat{Z}_t = \rho \hat{Z}_{t-1} + \epsilon_t, \tag{5}$$

where the innovations ϵ_t are normally distributed with zero mean and variance σ^2. The factor market equilibrium conditions are:

$$w_t = (1-\alpha) Z_t k_{t,2}^{\alpha}, \tag{6a}$$

$$r_t = \alpha Z_t k_{t,2}^{\alpha-1}. \tag{6b}$$

2.2 Temporary Equilibria

The first-order conditions to problem (3) can be reduced to the following equations:

$$\frac{1}{c_{t,1}} = \lambda_{t,1}, \tag{7a}$$

$$c_{t,1} = w_t - k_{t+1,2}, \tag{7b}$$

$$\lambda_{t,1} = \beta \mathbb{E}_t \frac{(1 - \delta + r_{t+1})}{c_{t+1,2}}, \tag{7c}$$

$$c_{t,2} = (1 - \delta + r_t) k_{t,2}, \tag{7d}$$

where $\lambda_{t,1}$ denotes the Lagrangian multiplier of the budget constraint $c_t - w_t - k_{t+1,2} = 0$. Together with the factor market equilibrium conditions (6) they determine a temporary equilibrium. It is easy to see that the sequence of temporary equilibria is governed by

$$k_{t+1,2} = \frac{\beta(1-\alpha)}{1+\beta} Z_t k_{t,2}^{\alpha}. \tag{8}$$

In the following, however, it will be instructive to analyze the model obtained from log-linearizing equations (6) and (7) at the stationary equilibrium.

2.3 Stationary Equilibrium

In the stationary equilibrium, the technology level $Z = 1$ is constant. In addition, $c_{t,1} = c_1$, $c_{t,2} = c_2$, and $k_{t,2} = k$. Equations (7a), (7c), and (6b), then, yield:

$$\frac{c_2}{c_1} = \beta(1 - \delta + \alpha k^{\alpha-1}). \tag{9}$$

[3] If Z_t follows a non-stationary process, e.g. a random walk, we need to transform the variables in the model so that the model is stationary in the transformed variables.

Since

$$c_1 = w - k = (1-\alpha)k^\alpha - k \tag{10}$$

and

$$c_2 = (1 - \delta + \alpha k^{\alpha-1})k$$

the stationary stock of capital is given by:

$$k = \left[\frac{1+\beta}{\beta(1-\alpha)}\right]^{\frac{1}{\alpha-1}}.$$

Of course, this is the stationary solution implied by (8). For illustrative purposes, let us pick values for the parameters α, β, δ, and ρ. We set $\alpha = 0.36$ equal to the capital income share in total production. As we consider two periods in our lifetime model, we look at a period length approximately equal to 30 years. Therefore, it is a reasonable assumption that capital has depreciated completely after one period, $\delta = 100\%$. From (7b), $\beta = 1/r$. Assuming that the annual real interest rate is equal to 4%, we should choose $r = (1.04)^{30} - 1 = 2.24$ implying $\beta = 0.446$. For the autocorrelation parameter of the productivity shock (5), we employ $\rho = 0.95$. This choice of parameters implies $k = 0.0792$, $c_1 = 0.178$, $c_2 = 0.145$, $w = 0.257$, and $r = 1.82$.

2.4 The Log-Linear Model

In the next step, we log-linearize all the equations (6) and (7) describing the temporary equilibrium of the model around the steady state. We will illustrate the procedure only for (7a).[4] To this end, take the logarithms of both sides of the equation (7a) and compute the total differential at the steady state:

$$-\frac{dc_{1,t}}{c_{1,t}} = \frac{d\lambda_{t,1}}{\lambda_{t,1}}.$$

Notice that $dc_{t,1} = c_{t,1} - c_1$. As a result, $-\hat{c}_{t,1} = \hat{\lambda}_{t,1}$. Similarly, equations (7) and (6) imply the following linear model:

$$-\hat{c}_{t,1} = \hat{\lambda}_{t,1}, \tag{11a}$$

[4] A detailed introduction to log-linearization can be found in Heer and Maußner (2009), Section 2.4.

$$\hat{c}_{t,2} = \hat{k}_{t,2} + \frac{rk}{c_2}\hat{r}_t, \tag{11b}$$

$$\hat{w}_t = \alpha\hat{k}_t + \hat{Z}_t, \tag{11c}$$

$$\hat{r}_t = (\alpha - 1)\hat{k}_t + \hat{Z}_t, \tag{11d}$$

$$\hat{k}_{t+1,2} = \frac{w}{k}\hat{w}_t - \frac{c_1}{k}\hat{c}_{t,1}, \tag{11e}$$

$$\hat{\lambda}_{t,1} = -\mathbb{E}_t\hat{c}_{t+1,2} + \beta\frac{c_1}{c_2}r\mathbb{E}_t\hat{r}_{t+1}. \tag{11f}$$

In the notation of (1), the variable $\mathbf{x}_t$ is equal to the capital stock, $\mathbf{u}_t$ consists of consumption $c_{t,1}$ and $c_{t,2}$, the wage rate w_t, and the real interest rate r_t. $\boldsymbol{\lambda}_t$ is equal $\lambda_{t,1}$, and $\mathbf{z}_t$ is simply the technology level Z_t.

We eliminate the wage and the interest rate from these system so that the following four equations result:

$$-\hat{c}_{t,1} = \hat{\lambda}_{t,1}, \tag{12a}$$

$$\hat{c}_{t,2} = \underbrace{\left[1 + \frac{(\alpha-1)r}{1-\delta+r}\right]}_{=:\Delta_1}\hat{k}_{t,2} + \frac{\Delta_1 - 1}{\alpha - 1}\hat{Z}_t, \tag{12b}$$

$$\hat{k}_{t+1,2} - \frac{\alpha w}{k}\hat{k}_{t,2} = -\frac{c_1}{k}\hat{c}_{t,1} + \frac{w}{k}\hat{Z}_t, \tag{12c}$$

$$(1-\Delta_1)\hat{k}_{t+1,2} + \hat{\lambda}_{t,1} = -\mathbb{E}_t\hat{c}_{t+1,2} + \frac{\Delta_1 - 1}{\alpha - 1}\mathbb{E}_t\hat{Z}_{t+1}. \tag{12d}$$

We can rewrite this system in the following way:

$$\begin{bmatrix}\hat{c}_{t,1}\\ \hat{c}_{t,2}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & -1\\ \Delta_1 & 0\end{bmatrix}}_{=:C_{x\lambda}}\begin{bmatrix}\hat{k}_{t,2}\\ \hat{\lambda}_{t,1}\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{\Delta_1-1}{\alpha-1}\end{bmatrix}}_{=:C_z}\hat{Z}_t, \tag{13}$$

and

$$\begin{aligned}&\underbrace{\begin{bmatrix}1 & 0\\ 1-\Delta_1 & 0\end{bmatrix}}_{=:D_{x\lambda}}\mathbb{E}_t\begin{bmatrix}\hat{k}_{t+1,2}\\ \hat{\lambda}_{t+1,1}\end{bmatrix} + \underbrace{\begin{bmatrix}-\frac{\alpha w}{k} & 0\\ 0 & 1\end{bmatrix}}_{=:F_{x\lambda}}\begin{bmatrix}\hat{k}_{t,2}\\ \hat{\lambda}_{t,1}\end{bmatrix}\\ &= \underbrace{\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}}_{=:D_u}\mathbb{E}_t\begin{bmatrix}\hat{c}_{t+1,1}\\ \hat{c}_{t+1,2}\end{bmatrix} + \underbrace{\begin{bmatrix}-\frac{c_1}{k} & 0\\ 0 & 0\end{bmatrix}}_{=:F_u}\begin{bmatrix}\hat{c}_{t,1}\\ \hat{c}_{t,2}\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{\Delta_1-1}{\alpha-1}\end{bmatrix}}_{=:D_z}\mathbb{E}_t\hat{Z}_{t+1} + \underbrace{\begin{bmatrix}\frac{w}{k}\\ 0\end{bmatrix}}_{=:F_z}\hat{Z}_t.\end{aligned} \tag{14}$$

Substituting (13) in (14) results in the following dynamic equation:

$$\underbrace{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}}_{=:B}\mathbb{E}_t\begin{bmatrix}\hat{k}_{t+1,2}\\ \hat{\lambda}_{t+1,1}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\alpha w}{k} & \frac{c_1}{k}\\ 0 & -1\end{bmatrix}}_{=:A}\begin{bmatrix}\hat{k}_{t,2}\\ \hat{\lambda}_{t,1}\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{w}{k}\\ 0\end{bmatrix}}_{=:C}\hat{Z}_t. \tag{15}$$

(15) corresponds to equation (2). In particular, the matrix B is singular and we cannot solve for the policy functions in the usual way.

The two lines of the linear system are two different equations in $k_{t+1,2}$. For both equations to hold simultaneously the two right hand sides of (15) must be equal, implying (making use of (10))

$$\hat{\lambda}_{t,1} = -\alpha\hat{k}_{t,2} - \hat{Z}_t = -0.36\hat{k}_{t,2} - \hat{Z}_t.$$

From this solution for $\hat{\lambda}_{t,1}$, we can compute the solution for $\hat{k}_{t+1,2}$ as follows:

$$\hat{k}_{t+1,2} = -\hat{\lambda}_{t,1} = 0.36\hat{k}_{t,2} + \hat{Z}_t.$$

Similarly, we can also compute the policy functions for consumption, wages, and the interest rate as functions of $\hat{k}_{t,2}$ and $\hat{Z}_t$ with the help of (11) and (12).

As a typical exercise in the business-cycle literature, one may assess the validity of the model as follows: 1) One estimates a time series for the (possibly autoregressive) process of the technology level Z_t, 2) one computes the time paths for the model variables such as consumption and investment with the help of the policy functions and the exogenous technology shocks, and 3) compares the computed time series with the corresponding empirical ones. As another typical exercise in the business-cycle literature, one may compute the second moments such as the standard deviation of the variables and their correlation with output that are implied by the theoretical model and compare them with those found in historical data. For a more detailed description see Heer and Maußner (2009), Section 1.5.

Of course, we do not use models with a period length of 20 or 30 years in a business-cycle analysis, but rather quarters or years. As one of the very few exceptions known to us, Brooks (2002) considers a model with four overlapping generations in order to study the effect of a demographic shock on stock and bond returns. A more complex business-cycle model with 60 overlapping generations is described in Section 10.2.1 in Heer and Maußner (2009).

In the next section, we will solve such singular systems as follows: 1) We calculate the Generalized Schur factorization of the matrices A and B and use them to write (15) as a nonsingular system. 2) We solve the transformed system in the two linear combinations of $(\hat{k}_{t,2}, \hat{\lambda}_{t,1})$, and 3), reverse this transformation to get the decision rules for $\hat{k}_{t+1,2}$ and $\hat{\lambda}_{t,1}$.[5]

[5] Another algorithm that is able to deal with a singular matrix B is described by King and Watson

3 The Generalized Schur Factorization

In this section, we describe the general method to solve the system (2) if the matrix B is singular. The Generalized Schur factorization of (A, B) representing the dynamic equations system (2) is given by

$$\begin{aligned} U^T B V &= S, \\ U^T A V &= T, \end{aligned} \tag{16}$$

where U and V are unitary matrices and S and T are upper triangular matrixes.[6] The eigenvalues of the matrix pencil are given by $\mu_i = t_{ii}/s_{ii}$ for $s_{ii} \neq 0$. Furthermore, the matrices S and T can be arranged so that the eigenvalues appear in ascending order with respect to their absolute value. We define new variables:

$$\begin{bmatrix} V_{xx} & V_{x\lambda} \\ V_{\lambda x} & V_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\boldsymbol{\lambda}}_t \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_t \\ \hat{\boldsymbol{\lambda}}_t \end{bmatrix}, \tag{17}$$

so that we can write (2) as

$$\begin{bmatrix} S_{xx} & S_{x\lambda} \\ 0 & S_{\lambda\lambda} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{x}}_{t+1} \\ \tilde{\boldsymbol{\lambda}}_{t+1} \end{bmatrix} = \begin{bmatrix} T_{xx} & T_{x\lambda} \\ 0 & T_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\boldsymbol{\lambda}}_t \end{bmatrix} + \underbrace{U^{-1}C}_{=:D} \hat{\mathbf{z}}_t. \tag{18}$$

In the following we denote by $n(x)$ the number of elements of the vector $x \in \mathbb{R}^n$ and assume that $n(x)$ eigenvalues have modulus less than one so that S_{xx} is a $n(x) \times n(x)$ upper triangular matrix, $S_{\lambda\lambda}$ is a $n(\lambda) \times n(\lambda)$ upper triangular matrix, and $S_{x\lambda}$ is a $n(x) \times n(\lambda)$ matrix. The matrices T_{xx}, $T_{\lambda\lambda}$, and $T_{x\lambda}$ have corresponding dimensions. In addition, we partition D into a $n(x) \times n(z)$ matrix D_x and the $n(\lambda) \times n(z)$ matrix D_λ:

$$D = \begin{bmatrix} D_x \\ D_\lambda \end{bmatrix}.$$

(2002). They apply singular value decompositions and QR factorizations to reduce the singular system to a non-singular one and then use the Schur (not the Generalized Schur!) factorization to solve the latter. Our algorithm uses the freely available Fortran routine `ZGGES` from the `LAPACK` package to get the Generalized Schur factorization. Thereby we defer the real cumbersome part of the system reduction to that program instead of having it to program ourselves as it is done by King and Watson (2002).

[6]See, e.g., Golub and van Loan (1996), Theorem 7.7.1, p. 377 who also describe the algorithm to compute the factorization of A and B. The set of all matrices of the form $A - \mu B$, $\mu \in \mathbb{C}$ is called a pencil. The eigenvalues of the pencil are the solutions of $|A - \mu B| = 0$. Unitary matrices U are complex-valued matrices whose conjugate transpose equals the inverse of U.

Given these assumptions and definitions the system

$$S_{\lambda\lambda}\mathbb{E}_t\tilde{\boldsymbol{\lambda}}_{t+1} = T_{\lambda\lambda}\tilde{\boldsymbol{\lambda}}_t + D_\lambda\hat{\mathbf{z}}_t \tag{19}$$

is unstable and has a forward solution:

$$\tilde{\boldsymbol{\lambda}}_t = \Phi\hat{\mathbf{z}}_t. \tag{20}$$

We construct the matrix Φ in a similar way as Heer and Maußner (2009), p.109-111. Consider the last line of (19):

$$s_{n(\lambda),n(\lambda)}\mathbb{E}_t\tilde{\lambda}_{n(\lambda),t+1} = t_{n(\lambda),n(\lambda)}\tilde{\lambda}_{n(\lambda),t} + \mathbf{d}'_{n(\lambda)}\hat{\mathbf{z}}_t, \tag{21}$$

and the last line of (20):

$$\tilde{\lambda}_{n(\lambda),t} = \boldsymbol{\phi}'_{n(\lambda)}\mathbf{z}_t, \tag{22}$$

where $\mathbf{d}'_{n(\lambda)}$ and $\phi'_{n(\lambda)}$ are the last row of D_λ and Φ, respectively. Since (1c) and (22) imply

$$\mathbb{E}_t\tilde{\lambda}_{n(\lambda),t+1} = \boldsymbol{\phi}'_{n(\lambda)}\Pi\mathbf{z}_t,$$

equation (21) can be rewritten as:

$$\left[s_{n(\lambda),n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)}\Pi - t_{n(\lambda),n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)} - \mathbf{d}'_{n(\lambda)}\right]\mathbf{z}_t = 0.$$

Therefore, the last row of Φ is given by

$$\boldsymbol{\phi}'_{n(\lambda)} = \mathbf{d}'_{n(\lambda)}\left(s_{n(\lambda),n(\lambda)}\Pi - t_{n(\lambda),n(\lambda)}I_{n(z)}\right)^{-1}.$$

Now, consider the next to last row of (19)

$$\begin{aligned} &s_{n(\lambda)-1,n(\lambda)-1}\mathbb{E}_t\tilde{\lambda}_{n(\lambda)-1,t+1} + s_{n(\lambda)-1,n(\lambda)}\mathbb{E}_t\tilde{\lambda}_{n(\lambda),t+1} \\ &\quad = t_{n(\lambda)-1,n(\lambda)-1}\tilde{\lambda}_{n(\lambda)-1,t} + t_{n(\lambda)-1,n(\lambda)}\tilde{\lambda}_{n(\lambda),t} + \mathbf{d}'_{n(\lambda)-1}\hat{\mathbf{z}}_t. \end{aligned} \tag{23}$$

Since

$$\tilde{\lambda}_{n(\lambda),t} = \boldsymbol{\phi}'_{n(\lambda)}\hat{\mathbf{z}}_t,$$

equation (1c) implies

$$\mathbb{E}_t\tilde{\lambda}_{n(\lambda),t+1} = \boldsymbol{\phi}'_{n(\lambda)}\Pi\hat{\mathbf{z}}_t,$$

and equation (23) can be reduced to

$$\begin{aligned}s_{n(\lambda)-1,n(\lambda)-1}\mathbb{E}_t\tilde{\lambda}_{n(\lambda)-1,t+1} &= t_{n(\lambda)-1,n(\lambda)-1}\tilde{\lambda}_{n(\lambda)-1,t} \\ &\quad + \left(t_{n(\lambda)-1,n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)} - s_{n(\lambda)-1,n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)}\Pi + \mathbf{d}'_{n(\lambda)-1}\right)\hat{\mathbf{z}}_t,\end{aligned}$$

with solution

$$\begin{aligned}\boldsymbol{\phi}'_{n(\lambda)-1} &= \left[t_{n(\lambda)-1,n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)} - s_{n(\lambda)-1,n(\lambda)}\boldsymbol{\phi}'_{n(\lambda)}\Pi + \mathbf{d}'_{n(\lambda)-1}\right] \\ &\quad \times \left(s_{n(\lambda)-1,n(\lambda)-1}\Pi - t_{n(\lambda)-1,n(\lambda)-1}I_{n(z)}\right)^{-1}.\end{aligned}$$

Proceeding in this way from the last line $i = n(\lambda)$ to the first line $i = 1$ of (19) we can compute the entire matrix Φ, where the i-th row of this matrix is:

$$\boldsymbol{\phi}'_i = \left[\mathbf{d}'_i + \sum_{j=i+1}^{n(\lambda)} \left(t_{i,j}\boldsymbol{\phi}'_j - s_{i,j}\boldsymbol{\phi}'_j\Pi\right)\right]\left(s_{i,i}\Pi - t_{i,i}I_{n(z)}\right)^{-1}.$$

To derive the solution for $\hat{\boldsymbol{\lambda}}_t$, consider the second line of equation (17):

$$V_{\lambda x}\tilde{\mathbf{x}}_t + V_{\lambda\lambda}\tilde{\boldsymbol{\lambda}}_t = \hat{\boldsymbol{\lambda}}_t.$$

Since the first line can be solved for $\tilde{\mathbf{x}}_t$,

$$\tilde{\mathbf{x}}_t = V_{xx}^{-1}\hat{\mathbf{x}}_t - V_{xx}^{-1}V_{x\lambda}\tilde{\boldsymbol{\lambda}}_t, \tag{25}$$

we find:

$$\hat{\boldsymbol{\lambda}}_t = V_{\lambda x}V_{xx}^{-1}\hat{\mathbf{x}}_t + \left[V_{\lambda\lambda} - V_{\lambda x}V_{xx}^{-1}V_{x\lambda}\right]\tilde{\boldsymbol{\lambda}}_t.$$

Inserting (20) yields:

$$\hat{\boldsymbol{\lambda}}_t = \underbrace{V_{\lambda x}V_{xx}^{-1}}_{=:L_x^\lambda}\hat{\mathbf{x}}_t + \underbrace{\left[V_{\lambda\lambda} - V_{\lambda x}V_{xx}^{-1}V_{x\lambda}\right]\Phi}_{=:L_z^\lambda}\hat{\mathbf{z}}_t. \tag{26}$$

To determine the policy function for the vector $\hat{\mathbf{x}}_{t+1}$, consider the first line of (18):

$$\begin{aligned}S_{xx}\tilde{\mathbf{x}}_{t+1} &= -\mathbb{E}_t S_{x\lambda}\tilde{\boldsymbol{\lambda}}_{t+1} + T_{xx}\tilde{\mathbf{x}}_t + T_{x\lambda}\tilde{\boldsymbol{\lambda}}_t + D_x\hat{\mathbf{z}}_t, \\ &= -S_{x\lambda}\Phi\Pi\hat{\mathbf{z}}_t + T_{xx}\tilde{\mathbf{x}}_t + T_{x\lambda}\tilde{\boldsymbol{\lambda}}_t + D_x\hat{\mathbf{z}}_t, \\ &= T_{xx}\tilde{\mathbf{x}}_t + \left(T_{x\lambda}\Phi - S_{x\lambda}\Phi\Pi + D_x\right)\hat{\mathbf{z}}_t.\end{aligned}$$

Since S_{xx} is invertible (since $n(x)$ of the eigenvalues of the pencil are within the unit circle), we can solve this equation for $\tilde{\mathbf{x}}_{t+1}$:

$$\tilde{\mathbf{x}}_{t+1} = S_{xx}^{-1}T_{xx}\tilde{\mathbf{x}}_t + S_{xx}^{-1}\left(T_{x\lambda}\Phi - S_{x\lambda}\Phi\Pi + D_x\right)\hat{\mathbf{z}}_t.$$

Since

$$\hat{\mathbf{x}}_{t+1} = V_{xx}\tilde{\mathbf{x}}_{t+1} + V_{x\lambda}\mathbb{E}_t\tilde{\boldsymbol{\lambda}}_{t+1} = V_{xx}\tilde{\mathbf{x}}_{t+1} + V_{x\lambda}\Phi\Pi\hat{\mathbf{z}}_t,$$

we get

$$\hat{\mathbf{x}}_{t+1} = V_{xx}S_{xx}^{-1}T_{xx}\tilde{\mathbf{x}}_t + \left\{V_{xx}S_{xx}^{-1}\left(T_{x\lambda}\Phi - S_{x\lambda}\Phi\Pi + D_x\right) + V_{x\lambda}\Phi\Pi\right\}\hat{\mathbf{z}}_t,$$

Substituting for (25) provides the policy function for the vector $\hat{\mathbf{x}}_{t+1}$:

$$\begin{aligned}\hat{\mathbf{x}}_{t+1} = {} & \underbrace{V_{xx}S_{xx}^{-1}T_{xx}V_{xx}^{-1}}_{=:L_x^x}\hat{\mathbf{x}}_t \\ & + \underbrace{\left\{V_{xx}S_{xx}^{-1}\left(T_{x\lambda}\Phi - S_{x\lambda}\Phi\Pi + D_x\right) + V_{x\lambda}\Phi\Pi - V_{xx}S_{xx}^{-1}T_{xx}V_{xx}^{-1}V_{x\lambda}\Phi\right\}}_{=:L_z^x}\hat{\mathbf{z}}_t\end{aligned} \tag{27}$$

Given the policy functions for $\hat{\boldsymbol{\lambda}}_t$, equation (1a) can be solved for the vector $\hat{\mathbf{u}}_t$:[7]

$$\hat{\mathbf{u}}_t = \underbrace{C_u^{-1}C_{x\lambda}\begin{bmatrix} I_{n(x)} \\ L_x^\lambda \end{bmatrix}}_{L_x^u}\hat{\mathbf{x}}_t + \underbrace{\left(C_u^{-1}C_{x\lambda}\begin{bmatrix} 0_{n(x)\times n(z)} \\ L_z^\lambda \end{bmatrix} + C_u^{-1}C_z\right)}_{L_z^u}\hat{\mathbf{z}}_t. \tag{28}$$

4 Implementation

We have implemented the algorithm of the previous section in the Gauss program `SolveLA3`, which can be downloaded from our web site.[8] The program takes the matrices C, D, F, and the matrix Π from (1) as input and returns the matrices of the policy functions L_j^i from the solution

$$\begin{aligned}\mathbf{x}_{t+1} &= L_x^x\mathbf{x}_t + L_z^x\mathbf{z}_t, \\ \mathbf{u}_t &= L_x^u\mathbf{x}_t + L_z^u\mathbf{z}_t, \\ \boldsymbol{\lambda}_t &= L_x^\lambda\mathbf{x}_t + L_z^\lambda\mathbf{z}_t.\end{aligned}$$

Applying this program to our example correctly delivers $L_x^x = 0.36$ and $L_z^x = 1$ as well as the coefficients of the policy functions for $\hat{\lambda}_{t,1}$, $\hat{c}_{t,1}$, and $\hat{c}_{t,2}$.

[7] See equation (2.63) in Heer and Maußner (2009), p. 113.

[8] The link is `http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/pap/gsf_prog.zip`.

5 Conclusion

In this article, we presented an algorithm for the computation of business cycle models that are described by singular linear (stochastic) difference systems. The method uses the Generalized Schur factorization and is easy to implement. It is also applicable to large-scale dynamic systems. For example in Heer and Maußner (2006), we used the algorithm to compute a monetary business-cycle model with more than 100 state variables.

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